Explicit modular formulae and symmetries of RCFT's. I

Antoine Coste
CNRS[*] Laboratory of Theoretical Physics
building 210, Paris University
91405 Orsay cedex, France
coste@math.uni-frankfurt.de

and

FB Math. Goethe Univ.
R. Mayer Strasse 8
D 60054 Frankfurt am Main, Germany

Abstract.

We derive compact formulae for modular transformation matrices of Wess-Zumino-Witten (WZW) affine characters. We start in this text which is conceived as the first of a series with the simple case of A_1 algebra at positive level $k = n - 2$, for which we can easily provide some description of isometry group and genus formula in a special case.

We also point to general features of these expressions, formulating and proving theorems for RCFT's which seem new.

1 Mathematical and Physical origin

Within the framework of 2d critical phenomena (so called "CFT's"), (our favourite primer is [1]), some positive Fourier series, called Virasoro characters encode critical exponents[3, 4], Casimir coefficients and all multiplicities of physical states. At first sight they are defined on the upper half plane, but since they have fascinating modular properties, identifying the Riemann surface on which they live and related symmetry groups is a progress. In this text we present some computations for $sl2$ WZW theories[2], but the general landscape is worth being observed:

After some years P. Bantay[9] proved W. Nahm's conjecture[5]: representations (which we

[*]UMR 8627

will generically call ρ in the sequel), of the modular group which occur in rational conformal theories, have principal congruence kernels, i.e. they are representations of some $SL_2(\mathbb{Z}/N\mathbb{Z})$. N is always the order of:

$$\rho(T) \ = \rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

As we show below, this theorem by Bantay brings a simplification in identifying the action of the modular group. In practice, this means replacing an $SL_2(\mathbb{Z})$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, by its residue matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $SL_2(\mathbb{Z}/N\mathbb{Z})$. Some nice formulae from "algebraic K-theory ", coming from the existence of inverses in the ring $\mathbb{Z}/N\mathbb{Z}$, and various related lemmas pave the way towards some compact formulae within a bounded number of steps. We therefore somehow improve on L. Jeffrey's seminal computations[17].

A consequence of this is also a better determination of the manifold of physical inequivalent τ's, a Riemann surface, which we propose to call **"dynamical moduli space"**, because multiplicity of excited physical states ("descendants") is a truely dynamical feature.

When $N = q_1\, q_2 \ldots\, q_r$, with each q_i being a prime power, decomposition of a $SL_2(\mathbb{Z}/N\mathbb{Z})$ group into its $SL_2(\mathbb{Z}/q\mathbb{Z})$ primary factors, (which means restriction of the rep. to the subgroups) can be performed as detailed in a former text: for each factor, the generators are, for each prime p_i dividing N,

$$T_i \ = T^{c_i}, \quad S_i \ = T^{1-c_i}\ ST^{1-c_i}\ ST^{1-c_i} S^{-1} \tag{1.1}$$

where we have used in the ring $\mathbb{Z}/N\mathbb{Z}$ the canonical decomposition into idempotents: $1 \equiv \sum c_i \mod N$; $c_i \equiv 1 \mod q_i$ and $\equiv 0$ mod the other q_j's. S and T are the commuting products of their primary images. This decomposition will be used in section 4.

We consider the representation ρ of dimension $n-1$ given by

$$S_{\alpha\beta} \ := \rho\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)_{\alpha\beta} = \sqrt{\frac{2}{n}} \ \sin\left(\frac{\pi\alpha\beta}{n}\right) \tag{1.2}$$

$$T_{\alpha\beta} \ := \rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)_{\alpha\beta} = \delta_{\alpha\beta} \quad e\left(\frac{\alpha^2}{4n} - \frac{1}{8}\right) \tag{1.3}$$

$$e(\tau) \ := \exp\left(2i\pi\tau\right)\ ,\ \alpha, \beta \in \{1, \ldots, n-1\},$$

$$\delta_{\alpha\beta} = \text{Kronecker's symbol with values } 0, 1$$

In this representation ρ, N, the order of $\rho(T)$, which we denoted n_∞ in our precedent texts, is therefore $4n$ if n is even, and $8n$ if n is odd.

Our main concern in the following is therefore to compute images of 2×2 matrices using their decomposition as words in S and T mod N.

2 $Gau\beta$ sums ingredients

Since T is diagonal, the technical point in computing the image of a matrix, which is a word in S and T, under representation ρ above, is being able to have adequate expressions for the

sum over β appearing in:

$$\rho(ST^C\ S)_{\alpha\gamma} = \sum_\beta S_{\alpha\beta}\ T^C_{\beta\beta}\ S_{\beta\gamma}$$

That is, the key ingredient is

$$\mathcal{T}(\alpha,\gamma,C,n) \quad := \quad \sum_{\beta=1}^{n-1} f(\beta) \tag{2.1}$$

$$\text{where} \quad f(\beta) \quad := \quad \sin\left(\frac{\pi\alpha\beta}{n}\right)\ \sin\left(\frac{\pi\beta\gamma}{n}\right)\ e\left(\frac{C\beta^2}{4n}\right) \tag{2.2}$$

Using the properties that: $f(\beta) = f(-\beta) = f(\beta+2n)$, and when $n|\beta$, $f(\beta)=0$; we can as well express $\mathcal{T}$ as a sum over $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}/2n\mathbb{Z}$, or $\mathbb{Z}/4n\mathbb{Z}$.

When $(C,2n)=1$ by completing the squares, we obtain:

$$\mathcal{T} = \frac{1}{8}\ S(C,4n)\ \left[e\left(\frac{-C^{-1}(\alpha-\gamma)^2}{4n}\right) - e\left(\frac{-C^{-1}(\alpha+\gamma)^2}{4n}\right)\right] \tag{2.3}$$

where C^{-1} is any integer k such that $4n|Ck-1$, in view of having unified notations, we always take in the following for C^{-1} the inverse of C mod 8n, rather than mod 4n.

$$S(C,N) := \sum_{\beta\in\mathbb{Z}/N\mathbb{Z}} e\left(\frac{C\beta^2}{N}\right) \quad \text{is a } \textit{Gau}\beta \textit{ sum.} \tag{2.4}$$

When $C = tn$ by splitting the sum between even and odd β's, one obtains:

$$\mathcal{T} = \frac{n}{4}\left[\left(1+i^{C/n}\ (-1)^{\frac{\alpha-\gamma}{n}}\right)\delta_{\alpha,\ \gamma\ modn} - \left(1+i^{C/n}\ (-1)^{\frac{\alpha+\gamma}{n}}\right)\delta_{\alpha,\ -\gamma\ modn}\right] \tag{2.5}$$

When $C = 2\Gamma$, $(\Gamma,n)=1$, we have to distinguish according to the parity of $\alpha-\gamma$: if even, the square is obviously completed in the sum as above, whereas if it is equal to some $2l+1$, we need an extra lemma, making a sum over $\mathbb{Z}/8n\mathbb{Z}$ enter the game:

$$\begin{aligned} \mathcal{T} &= \frac{i}{2}\ e\left(\frac{\Gamma^{-1}\ (\alpha^2\ +\gamma^2)}{8n}\right)\ sin\left(\frac{\pi\Gamma^{-1}\ \alpha\ \gamma}{2n}\right)\ SF \\ SF &= S(\Gamma,2n) \qquad \text{for } \alpha-\gamma \text{ even} && (2.6) \\ SF &= \frac{1}{2}\ S(\Gamma,8n)\ - S(\Gamma,2n) \ \ \text{for } \alpha-\gamma \text{ odd} && (2.7) \end{aligned}$$

3 Explicit compact expressions

When $(C,2n)=1$ set $U := (A+1)C^{-1},\ V := (D+1)C^{-1}$, $\zeta_8 = e(1/8)$.

$$\rho\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)_{\alpha\lambda} = \rho\left(T^U\ ST^C\ ST^V\ \right)_{\alpha\lambda} \tag{3.1}$$

$$= \frac{2}{n}\ \zeta_8^{-2-U-V}\mathcal{T}(\alpha,\gamma,C,n)\ e\left(\frac{\alpha^2\ U+\lambda^2\ V}{4n}\right)$$

$$= \frac{1}{2n}\ \zeta_8^{2-C-U-V}\ S(C,4n) \qquad \sin\left(\frac{\pi C^{-1}\ \alpha\lambda}{n}\right) e\left(\frac{C^{-1}\ (\alpha^2 A+\lambda^2 D)}{4n}\right) \tag{3.2}$$

Note that for $(C,2n)=1,\ S(C,4n)$ is a Galois conjuguate of the famous Dirichlet's $S(1,4n)=2\sqrt{n}\ (1+i)$, therefore is never zero; the matrix element for $\alpha=1,\ \lambda=2$ doesn't vanish either. Assuming known congruence results for Jacobi forms , or Bantay's general theorem, we have therefore proven:

Proposition: consider four integers satisfying $ad-bc=1$; for the representation ρ coming from affine Lie algebra sl_2 at level $k=n-2$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Ker\ \rho \qquad \text{implies that } (c,\ 2n)\neq 1 \tag{3.3}$$

When $(c,2n)=1$ **and** n **odd** , we can simplify the above expression using Legendre's symbol (which takes here only values 1 or -1):

$$S(c,4n)=2(1+i^{nc})\ \left(\frac{c}{n}\right) S(1,n) \tag{3.4}$$

Since mod 8, any odd residue satisfies, $C\equiv C^{-1}$, we can recast the eighth root of unity factors into the form (which is a step towards a formula in next section):

$$\rho\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)_{\alpha\lambda} =$$

$$= \sqrt{\frac{2}{n}}\ \left(\frac{C}{n}\right)\ \zeta_8^{g(C,n)-(A+D+3)C}\ \sin\left(\frac{\pi C^{-1}\ \alpha\lambda}{n}\right) e\left(\frac{C^{-1}\ (\alpha^2 A+\lambda^2 D)}{4n}\right) \tag{3.5}$$

where $g(C,n)$ is an integer which depends only on the residues of c and n mod 4:

$$\begin{aligned} &\text{if} \quad C\equiv 1\ mod\ 4, \quad g=3 \\ &\text{if } C\equiv -1 \text{ and } n\equiv 1\ mod\ 4, \quad g=1 \\ &\text{if } C\equiv -1 \text{ and } n\equiv -1\ mod\ 4, \quad g=-3 \end{aligned} \tag{3.6}$$

A first check of this formula is that it gives the same image for a matrix and its opposite, this indeed results from properties of Legendre's symbol and from:

$$g(C,n)-g(-C,n)+2C\ \equiv\ 2(n+1)\ mod\ 8 \tag{3.7}$$

When $(d,2n)=1$, denoting as above by D^{-1} the inverse mod $8n$ and $X:=-(C+1)D^{-1}$, $Y:=(B-1)D^{-1}$ we get:

$$\rho\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)_{\alpha,\lambda} = \rho(T^X\ ST^{-D}\ ST^Y S)_{\alpha,\lambda}$$

$$= \left(\frac{2}{n}\right)^{3/2} \frac{1}{4}\ \zeta_8^{D-X-Y-2} S(-D,4n) e\left(\frac{\alpha^2 BD^{-1}}{4n}\right)\ \mathcal{T}(\alpha D^{-1},\lambda,-CD^{-1},n) \tag{3.8}$$

This gives in particular, using above results (δ symbols mod n can be recast into equalities mod 2n), a decorated Galois permutation (notice $A\equiv D^{-1}$ mod N):

$$\begin{aligned}\rho\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right)_{\alpha,\lambda} &= \frac{1}{\sqrt{8n}}\ \zeta_8^{D-X-Y-2}\ S(-D,4n)\ e\left(\frac{AB\alpha^2}{4n}\right) \\ &\cdot \quad [\delta_{\alpha D^{-1},\ \lambda\ mod\ 2n} - \delta_{\alpha D^{-1},\ -\lambda\ mod\ 2n}] \end{aligned} \tag{3.9}$$

$$= \pm \zeta_8^{2(A-1)-AB} e\left(\frac{AB\alpha^2}{4n}\right) [\delta_{A\alpha,\ \lambda\ mod\ 2n} - \delta_{A\alpha,\ -\lambda\ mod\ 2n}] \tag{3.10}$$

where the sign is studied in [15].

4 Galois properties

C. Itzykson and J. Lascoux early recognized the relevance of classical Galois theory for study of CFT's. Here let us comment on the "etat de l'art", using the simplicity of the matrices involved, allowing short trigonometric expressions:

When d and n are >0 , and $(d,2n)=1$, quadratic reciprocity law applied to odd factors of n gives:

$$\begin{aligned}\sigma_d\ (S_{\alpha,\beta}) &= \left(\frac{-2n}{d}\right)\sqrt{\frac{2}{n}}\ \sin\left(\frac{\pi d\alpha\beta}{n}\right) &(4.1)\\ &= \left(\frac{-2n}{d}\right)\ sign(n- <\alpha d>_{2n})\ S_{\sigma_d(\alpha),\beta} &(4.2)\end{aligned}$$

where $<u>_{2n}$ is the number $\in\{0,\ldots,2n-1\}$ congruent to u mod $2n$. Here we do not suppose n odd, notice that if an odd power of 2 enters into the decomposition of n, $\left(\frac{4}{d}\right)=1$ and there is no $\sqrt{2}$ in the formula.

$$\begin{aligned}\text{if } \alpha d = 2ln+\gamma \quad \sigma_d\ (\alpha) &:= \gamma \\ \text{whereas if } \alpha d=(2l+1)n+\gamma \quad \sigma_d\ (\alpha) &:= n-\gamma\end{aligned} \tag{4.3}$$

Proposition: When L is invertible in $\mathbb{Z}/8n\mathbb{Z}$, the cyclotomic action on the image of **any** matrix is†:

$$\sigma_L\left(\rho\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)\right) = \rho\left(\begin{pmatrix} A & BL \\ CL^{-1} & D \end{pmatrix}\right) \tag{4.4}$$

† a.c. thanks M. Bauer for, a long time ago, pointing this

Proof: Since left and right hand sides are group morphisms and that SL_2 is generated by T and S , it suffices to check it for T, this is obvious, and for S: Look therefore at the explicit expressions obtained for the image of a matrix with $C = 1$, or even C invertible, distinguishing again the cases, we see another little miracle for n odd:

$$\left(\frac{C}{n}\right)\ \zeta_8^{g(C,n)\ -3C} = \left(\frac{-2n}{C^{-1}}\right) = \left(\frac{-2n}{C}\right) = (-1)^{\frac{c^2-1}{8}+\frac{c-1}{2}\frac{n+1}{2}}\ \left(\frac{C}{n}\right) \tag{4.5}$$

Which is just what is needed to insure equality of the directly computed matrix and the Galois image of $\rho(S)$.

Finally for any invertible C mod $8n$:

$$\rho\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)_{\alpha\lambda} = \sigma_{C^{-1}}\left(\rho\left(\begin{pmatrix} A & BC \\ 1 & D \end{pmatrix}\right)_{\alpha\lambda}\right) = \tag{4.6}$$

$$\rho(T^{(A+1)C^{-1}}\ ST^{C}\ ST^{(D+1)C^{-1}}\ )_{\alpha\lambda} = \sigma_{C^{-1}}(\rho(T^{A}\ ST^{D})_{\alpha\lambda}) \tag{4.7}$$

$$= \left(\frac{-2n}{C}\right)\ \sqrt{\tfrac{2}{n}}\ \ \sin\left(\frac{\pi C^{-1}\alpha\lambda}{n}\right)\ e\left(\frac{C^{-1}\ (A\alpha^2 + D\lambda^2)}{4n}\right) \tag{4.8}$$

In [9] P. Bantay gave a very interesting criterion for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $(d, 2n) = 1$, to be in the kernel. It is related to generation properties of finite unimodular matrix groups [21, 22]. Although we will have much more to say in later studies, we can confirm that the above formula inserted in that criterion and the direct determination of the kernel for some values of n agree with our direct investigations which we begin here to present in section 6.

Explicitly, $\sigma_d\ (S)T^b = T^c\ S$ reads:

$$\left(\frac{-2n}{d}\right)\sin\left(\frac{\pi d\alpha\beta}{n}\right) e\left(\frac{\beta^2 b - \alpha^2 c}{4n}\right)\zeta_8^{c-b} = \sin\left(\frac{\pi\alpha\beta}{n}\right) \tag{4.9}$$

for all $\alpha, \beta = 1, \ldots, n-1$. Taking the norm gives

$$\sin\left(\frac{\pi\alpha\beta(d-1)}{n}\right)\ \sin\left(\frac{\pi\alpha\beta(d+1)}{n}\right) = 0$$

This implies that if $(d, 2n) = 1$, $d = nL + d_0$ with $d_0 = \pm 1$. Then

$$\sin\left(\frac{\pi d\alpha\beta}{n}\right) = d_0(-1)^{L\alpha\beta}\sin\left(\frac{\pi\alpha\beta}{n}\right)$$

Therefore, for n bigger than 4, $L = 2l$, and we distinguish:

For n **odd**, above eqs. for all α, β require a sign equality

$$b = 4nb'\ \ , c = 4nc'\ \ , (-1)^{b'-c'} = \varepsilon(d\ , n) \tag{4.10}$$

where one can, distinguishing the values of d_0, reshuffle this into:

$$(-1)^{b'-c'} = \varepsilon(2ln + d_0\ , n) = (-1)^{\frac{l\ (\ l\ - d_0)}{2}} \tag{4.11}$$

We therefore obtain the following expression for the Bantay's criterion:

The matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which have $(d, 2n) = 1$, and are in the kernel, are for n odd, those congruent mod $8n$ to :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 4n \\ 4n & 1 \end{pmatrix}, \begin{pmatrix} 2n+1 & 0 \\ 0 & 2n+1 \end{pmatrix}, \begin{pmatrix} 2n+1 & 4n \\ 4n & 2n+1 \end{pmatrix},$$
$$\begin{pmatrix} 2n-1 & 4n \\ 0 & 2n-1 \end{pmatrix} , \begin{pmatrix} 2n-1 & 0 \\ 4n & 2n-1 \end{pmatrix}, \begin{pmatrix} 4n+1 & 0 \\ 4n & 4n+1 \end{pmatrix}, \begin{pmatrix} 4n+1 & 4n \\ 0 & 4n+1 \end{pmatrix},$$

and their opposite.

For n **even** we find that the same criterion picks matrices congruent **mod** $4n$ to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2n+1 & 0 \\ 0 & 2n+1 \end{pmatrix},$$

and their opposite. We will comment elsewhere, in collaboration with P. Bantay and friends, we hope, on this criterion, which applies to conjugacy classes, and its relationship with the enumeration of $SL_2(\mathbb{Z}/N\mathbb{Z})$ we have sketched.

5 A general theorem

In contradistinction with the first sections, where the computations where self contained we now rely fully on established general results, in particular the theorem that the modular representation is defined in a cyclotomic field $\mathbb{Q}(\zeta_M)$. It has been proven[9] that M equals the conductor N. It would anyway be no trouble to enlarge the field if needed for further purpose. We therefore have at disposal the Galois morphisms which we denote as always by σ_L . If we were reasoning with $c \in \mathbb{Z}$, with $ch - qM = 1$, we could use the absolute Galois morphism $\sigma_h = \sigma_{C^{-1}}$ here. The way we successfully tackled the eighth roots of unity, going from n to $8n$ when needed in the first sections illustrate this.

To make the notations lighter, we drop in this section the reference to ρ , calling S and T the images by ρ of the $SL_2(\mathbb{Z})$ matrices defined in the beginning, as Bantay does.

Theorem 1: In any RCFT, if N is the order of T, and C invertible mod N,

$$\begin{aligned} \rho\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) &= \sigma_{C^{-1}}(\ T^A\ ST^D\ ) \\ \text{where} \quad \sigma_{C^{-1}} &\in Gal(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \end{aligned} \tag{5.1}$$

Proof:

$\sigma_{C^{-1}}(T^A\ ST^D) = T^{AC^{-1}}\ \sigma_{C^{-1}}(S)\ T^{DC^{-1}}$.

But relying on theorems proven by T. Gannon, J. Lascoux and the author, Bantay has established [9] ‡ that:

$$\sigma_{C^{-1}}(S) \ = T^{C^{-1}}\ ST^C\ ST^{C^{-1}} \tag{5.2}$$

‡Of course many people contributed significantly to the development of this field: J. de Boer, C. Goeree, C. Itzykson, W. Nahm, J.B. Zuber, V. Pasquier, P. Ruelle, E. Thiran, J. Wyers, D. Altschuler, M. Bauer, ...

Therefore

$$\sigma_{C^{-1}}(T^A S T^D) = T^{AC^{-1}+C^{-1}} S T^C S T^{C^{-1}+DC^{-1}} \ = \rho\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)$$

Theorem 2:

In any RCFT, where N, the order of $\rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, doesn't divide 12,

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Ker\ \rho$, **Then** $(c, N) \neq 1$.

Proof : If $(c, N) = 1$, one would have $\sigma_{C^{-1}}(T^a S T^d) = Id$, therefore $S^{-1} \ = T^{a+d}$.Thus S and T commute, which implies $S^{-1} \ = T^3$.

Therefore N divides $a + d - 3$, $4(a + d)$, and $3(a + d + 1)$.

If $a+d$ is even , N is odd since it divides $a+d-3$, but since it divides $4(a+d)$, it should divides both $a + d$ and 3.

If $a + d = 3$, N divides 12. If $a + d \neq 3$ is odd , N is even since it divides $a + d - 3$, but then it also divides $a + d + 1$, thus it divides 4.

6 A genus formula

Proposition: when $n \equiv 3\ mod\ 4$ characters of affine $sl2$ algebra at level $k = n - 2$, bring a representation of $SL_2(\mathbb{Z}/8n\mathbb{Z})$ which is separately a representation of $SL_2(\mathbb{Z}/8\mathbb{Z})/\pm 1$ and of $SL_2(\mathbb{Z}/n\mathbb{Z})/\pm 1$. The kernel of the first one is exactly

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 4 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 4 & 5 \end{pmatrix}\} \text{ in } \ SL_2(\mathbb{Z}/8\mathbb{Z})/\pm 1 \tag{6.1}$$

Proof: when n is odd, $1 = c_2 + c_n$, with $c_2 = n^2$, $c_n = 1 - n^2$ is a decomposition into idempotents, because any odd number is a square root of 1 mod 8. The restriction of the representation to the $SL_2(\mathbb{Z}/8\mathbb{Z})$ subgroup is therefore given by images of S_2 and $T_2 = T^{n^2}$ as explained in section 1.

$$\rho(T_2)_{\alpha,\lambda} \ = \delta_{\alpha,\lambda}\ e\left(\frac{n\alpha^2}{4} - \frac{1}{8}\right) \tag{6.2}$$

$$\text{implies that} \quad \rho(T_2)^4 \ = \ -Id_{\mathbb{C}^{n-1}} \text{ is central} \tag{6.3}$$

$$\text{since} \quad S_2 \equiv \begin{pmatrix} 1-n^2 & -n^2 \\ n^2 & 1-n^2 \end{pmatrix}\ ,\ S_2^2 \equiv \begin{pmatrix} 1-2n^2 & 0 \\ 0 & 1-2n^2 \end{pmatrix} \in SL_2(\mathbb{Z}/8n\mathbb{Z})\ , \tag{6.4}$$

for $n \equiv 3\ mod\ 4$ the formulae of section 3 above give $\rho(S_2^2) = Id$. This with centrality of $\rho(T_2)^4$, is sufficient to prove that the four above matrices (which are words in S_2 , T_2) are in the kernel. We have already discussed in detail the conjugacy classes structure of $SL_2(\mathbb{Z}/8\mathbb{Z})/\pm 1$, a group with 192 elements, in a previous work [16]. Computer packages can also give a lot of useful outputs for cross-checks. Here the cardinal of the image of this group should divides $192/4 = 48$, and should be a multiple of 8, the order of $\rho(T_2)$. This requirement implies that the kernel can only be the normal subgroup given by the four matrices given above.

Let us comment an example: applying Bantay's criterion we find that for n odd

$$\begin{pmatrix} 2n+1 & 4n \\ 4n & 2n+1 \end{pmatrix} \in Ker\rho \bigcap SL_2(\mathbb{Z}/8n\mathbb{Z})$$

This matrix is $\equiv \begin{pmatrix} 2n+1 & 4 \\ 4 & 2n+1 \end{pmatrix}$ mod 8. For $n \equiv 3$ mod 4, which is the case of our prop. above, it is $\equiv \begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix}$ which is one of the element of the kernel we identified in $SL_2(\mathbb{Z}/8\mathbb{Z})/\pm 1$.

Proposition: for $n = p$ a prime bigger or equal to 7 and congruent to 7 mod 4 , the genus of the completed (smooth, compact, without punctures) Riemann surface, on which characters live, is:

$$\begin{aligned} g &= 1 + 12\, p(p^2 -1) \left(\frac{1}{6} - \frac{1}{8p}\right) \\ = 1 + \frac{(p^2 -1)(4p-3)}{2} &= \frac{4p^3 -3p^2 -4p+5}{2} \end{aligned} \tag{6.5}$$

To make the proof obvious let us first notice:

lemma: Let ρ be a representation of $G = G_1 \times G_2$, a direct product, denote the projection by:

$$\begin{array}{ccc} Ker\ \rho & \longrightarrow & G_1 \times G_2 \\ g & \longrightarrow & (\varphi_1(g),\ \varphi_2(g)) = (g_1,\ g_2) \end{array}$$

Then $\varphi_2(Ker\rho) = \{g_2,$ for which exists $g_1, \rho(g_1 g_2) = 1\}$ is a normal subgroup of G_2 . This is trivial:

$$\rho(hg_1h^{-1}\ hg_2h^{-1}) = 1$$

Now, excepting primes 2 and 3, $SL_2(\mathbb{Z}/p\mathbb{Z})/\pm 1$ is simple and of order $p(p^2 -1)/2$, Therefore we showed above that when $n = p \equiv 7\ mod 4$, the image of the kernel in $SL_2(\mathbb{Z}/p\mathbb{Z})$ is exactly $\pm Id$. therefore $Im\ \rho$ is a group with exactly $48\ \times \frac{p(p^2 -1)}{2}$ elements. Expression of the genus is given by Riemann-Hurwitz formula for a triangulation with $8p$-valent vertices. For $p = 7$, $g = 601$, which is big, but $g - 1$ is already eight times smaller than the corresponding value for principal curve $X(8p)$. When n factors into many powers, each $\rho(S_i)^2$ is a central element of the representation, not necessarily a constant.

7 Appendix

Here we give some explicit expressions of characters for the interested reader. First we have the useful expansions:

$$\frac{1}{\prod(1-q^n)^3} = 1 + 3q + 9q^2 + 22q^3 + 51q^4 + 108q^5 + 221q^6 + 429q^7 + 810q^8 + \dots$$

$$
\begin{aligned}
-\,ln(\prod(1-q^n)\ ) &= \sum_{k>0} \frac{\sigma_1(k)\ q^k}{k} \\
&= \frac{q}{1-q} + \sum_{p\ prime} \frac{q^p}{p} + \frac{3}{4}q^4 + q^6 + \frac{7}{8}q^8 + \dots
\end{aligned}
$$

$$
\text{where}\ \ \sigma_1(k = \prod p^\nu) = \sum_{m|k} m = \prod \frac{(p^{\nu+1} - 1)}{(p-1)}
$$

The formula for sl_2 characters labelled by shifted weights $\lambda = 1, \dots, n-1$ is[7, 1]:

$$
\chi_{\lambda,[n]} = \frac{1}{\eta^3} \sum_{x \equiv \lambda\ mod\ 2n} x\ q^{\frac{x^2}{4n}} \tag{7.1}
$$

We have been concerned with the representation underlying the identity:

$$
\chi_\alpha\left(\frac{-1}{\tau}\right) = \sum_{\beta=1}^{n-1} \rho(S)_{\alpha,\beta}\ \chi_\beta(\tau) \tag{7.2}
$$

Let us focus on the case $n = 3$ (i. e. $k = 1$), the two characters are:

$$
\begin{aligned}
\chi_1 &= q^{\frac{-1}{24}} \frac{\sum_{l\in\mathbb{Z}} (1+6l)\ q^{l(1+3l)}}{\prod(1-q^n)^3} \\
&= q^{\frac{-1}{24}} (1 + 3q + 4q^2 + 7q^3 + 13q^4 + 19q^5 + 29q^6 + 43q^7 + 62q^8 + 90q^9 + \dots) && (7.3) \\
\chi_2 &= 2\ q^{\frac{5}{24}} \frac{\sum_{l\in\mathbb{Z}} (1+3l)\ q^{l(2+3l)}}{\prod(1-q^n)^3} \\
&= q^{\frac{5}{24}} (2 + 2q + 6q^2 + 8q^3 + 14q^4 + 20q^5 + 34q^6 + 46q^7 + 70q^8 + 96q^9 + \dots) && (7.4)
\end{aligned}
$$

They satisfy

$$
\chi_1\chi_2\ (\chi_1^4 - \chi_2^4) = 2 \tag{7.5}
$$

Therefore our "dynamical moduli space" is here, when we adopt the χ's as coordinates, a smooth complex curve, of genus 10, degree 6, here embedded in projective space $\mathbb{P}^2$. It has six points at infinity.

Since the isometry group, which is a Galois group for an extension of the function field $\mathbb{Q}(j)$ is here solvable, we can solve the above equation by radicals, which means we can parametrize our "dynamical moduli space" in terms of a single complex number t:

$$
\text{set}\ t := \chi_1\chi_2\ \ \text{then}\ \chi_1^8 - \frac{2}{t}\chi_1^4 - t^4 = 0 \tag{7.6}
$$

$$
\begin{aligned}
\chi_1 &= i^\alpha\ t^{\frac{-1}{4}}\ (1 \pm \sqrt{1+t^6})^{\frac{1}{4}} \\
\chi_2 &= i^{-\alpha}\ t^{\frac{5}{4}}\ (1 \pm \sqrt{1+t^6})^{\frac{-1}{4}} && (7.7)
\end{aligned}
$$

The already high genus, is due to the cuts needed in taking roots. This case is exceptional in the sense that for higher n's the $j(\tau)$ function enters the game.

Acknowledgements:

Without kind personal support of J. Wolfart, this work wouldn't have been possible. We'd like also to thank for hospitality or nice scientific conversations: H. Behr, M. Pflaum und die Fachbereich Math. und Rechnenbetrieb Leute der Goethe Univ. , M. Lüscher, R. Stora, P. de la Harpe, A. Alekseev,J. Gasser, P. Minkowski, einige Fachleute aus Zürich und Ihes, J.P. Derendinger, D. Altschuler, J. Lascoux, M. Giusti, M. Streit, G. Kemper, E. Cremmer, B. Doucot, J. Magnen, P. Viot, V.Pasquier.

References

[1] C. Itzykson, J.M. Drouffe, "Statistical Theory of Fields", two volumes, Cnrs editions, (1989).

[2] E. Witten, "Non-Abelian bosonization..." Comm. Math. Phys. 92, (1984) 455-472.

[3] A. Belavin, A. Polyakov, A. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.

[4] M. Luescher, G. Mack Commun. Math. Phys. 41 (1975) 203.

[5] W. Nahm, Comm. Math. Phys. 118 (1988) 171-176.

[6] J. Cardy, Nucl. Phys.B 270, (1986) 186-204.

[7] V. Kac, "Infinite Dimensional Lie algebras", esp. chap 13, third ed. Cambridge Univ. press, paperback (1990).

[8] D. Zagier, M. Eichler, "Jacobi Forms" Birkhauser. D. Zagier, cours au College de France, Paris und Bonn.

[9] P. Bantay, "The kernel of the modular representation and the Galois action in RCFT",Commun.Math.Phys. 233 (2003) 423-438.

[10] T. Kawai, "On the structure of fusion algebras", Phys. Letters B 217 (1989), 247-251.

[11] C. Itzykson et al., "Comments on the links between su(3) invariants, Jacobian of the Fermat curve....." J. of Geometry and Physics 22, 134-189.

[12] A. Coste, T. Gannon, Phys. Letters B 323, 316-321. P. Ruelle, E. Thiran, J. Wyers," Implications of an arithmetical symmetry of the commutant...."Nucl. Phys. B402 (1993), 693-708.

[13] J. de Boer, J. Goeree, Comm. Math. Phys. 139,267 (1991).

[14] W. Eholzer,Commun.Math.Phys 172, 623 (1995), unpublished. W. Eholzer, N.-P. Skoruppa, Commun.Math.Phys 174, 117 (1995).

[15] A. Coste, P. Degiovanni, J. Lascoux et al, "Precise study of some number fields and Galois actions...", Ann. Inst. H. Poincare Paris, vol 63 nb 1, 41-79. esp. paragraph 2 and apend. A.

[16] A. Coste, "Investigations sur les caracteres de Kac-Moody...", Ihes preprint , publications in preparation.

[17] L. Jeffrey, Comm. Math. Phys. 147, (1992), 563-604.

[18] Rademacher, "Analytische Zahlentheorie", Grundlehren , Band 169, Springer (1973). E. Hecke, "Zahlentheorie".

[19] J. Wolfart "Einfuehrung in die Zahlentheorie und Algebra" F. Vieweg und Sohn Verlagsgesellschaft, Aufbaukurs Mathematik, Braunschweig Wiesbaden (1996).

[20] J. Wolfart, Manuscripta Math. 17 (1975) 339-362. J. Wolfart, A. Nobs, Comment. Math. Helv. 39 (51)(1976) 491-526, Math. Zeitschrift 138 (1974) 239-254.

[21] H. Behr, W. Eholzer, T. Gannon, private communications.

[22] P. de la Harpe, "Topics in Geometric Group Theory", Chicago Univ. Press and refs Therein. P. de la Harpe and A. Valette, priv. comm..

[23] G. Toulouse, "Les scientifiques et les droits de l'Homme", ed. de la Maison des sciences de l'homme, Paris 2003.